Also solved by Dionne Bailey, Elsie Campbell,and Charles Diminnie, Angelo State University, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Hyun Bin Yoo, South Korea and the proposer.

- **5620:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania*

Prove: If $a, b, \in [0,1]$; $a \le b$, then

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \le 2(a+b).$$

Solution 1 by Moti Levy, Rehovot, Israel

Let

$$\alpha := \sqrt{ab} \le 1, \quad \beta := \frac{a+b}{2} \le 1, \quad r := \frac{b}{a},$$

Then the original inequality can be reformulated as

$$\sqrt{r} \le \frac{r^{\alpha} + r^{\beta} + r^{1-\alpha} + r^{1-\beta}}{4} \le \frac{1}{2} + r.$$

Since $f(x) := r^x$ is convex function, then

$$\frac{r^{\alpha} + r^{\beta} + r^{1-\alpha} + r^{1-\beta}}{4} \ge r^{\frac{\alpha+\beta+(1-\alpha)+(1-\beta)}{4}} = \sqrt{r}.$$

The Bernoulli's inequality is

$$(1+x)^{\alpha} \le 1 + \alpha x, \quad 0 \le \alpha \le 1, \quad x \ge -1.$$

Using the Bernoulli's inequality we get

$$r^{\alpha} \le 1 + \alpha(r-1),$$
$$r^{1-\alpha} \le 1 + (1-\alpha)(r-1)$$

hence

$$r^{\alpha} + r^{1-\alpha} \le 1 + r.$$

It follows that

$$\frac{\left(r^{\alpha} + r^{1-\alpha}\right) + \left(r^{\beta} + r^{1-\beta}\right)}{4} \le \frac{1}{2} + r.$$

Remark: The constraint $a \le b$ is redundant.

Solution 2 by Michel Bataille, Rouen, France

We suppose $a, b \in (0,1]$ and do not use the hypothesis $a \le b$.

Let

$$M = a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right).$$

Since $\sqrt{ab}$ and $\frac{a+b}{2}$ are in $(0,1]$, the functions $x \mapsto x^{\sqrt{ab}}$ and $x \mapsto x^{\frac{a+b}{2}}$ are concave on $(0,\infty)$. It follows that

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \le (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\sqrt{ab}} = a+b$$

and

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\frac{a+b}{2}} = a+b.$$

By addition, $M \le 2(a+b)$.
If m is a positive real number, the function $x \mapsto m^x$ is convex on R. Taking successively $m = \frac{b}{a}$ and $m = \frac{a}{b}$ and setting $k = \frac{1}{2}\left(\sqrt{ab} + \frac{a+b}{2}\right)$, it follows that

$$\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{b}{a}\right)^{k}$$

and

$$\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{a}{b}\right)^{k}.$$

Using $x+y \ge 2\sqrt{xy}$ for positive x, y, we deduce that

$$M \ge 2\left(a\left(\frac{b}{a}\right)^{k} + b\left(\frac{a}{b}\right)^{k}\right) \ge 2\cdot 2\left(a\left(\frac{b}{a}\right)^{k}\cdot b\left(\frac{a}{b}\right)^{k}\right)^{1/2}$$

and $M \ge 4\sqrt{ab}$ follows.

Solution 3 by Arkady Alt, San Jose, California

Applying inequality $x+y \ge 2\sqrt{xy}, x,y>0$ to $(x,y) = \left(a\left(\frac{b}{a}\right)^{\sqrt{ab}}, b\left(\frac{a}{b}\right)^{\sqrt{ab}}\right)$ and to $(x,y) = \left(a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}, b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right)$ we obtain

$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \ge 2\sqrt{a\left(\frac{b}{a}\right)^{\sqrt{ab}}\cdot b\left(\frac{a}{b}\right)^{\sqrt{ab}}} = 2\sqrt{ab}$ and

$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\sqrt{a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}\cdot b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}} = 2\sqrt{ab}.$

Thus, $a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \ge 4\sqrt{ab}.$

For function $f(t) = t^p$, which for $p \in [0,1]$ is concave down on $(0,\infty)$, holds inequality $\dfrac{ax^p + by^p}{a+b} \le \left(\dfrac{ax+by}{a+b}\right)^p$ for any $x, y > 0$.

Since $\sqrt{ab}, \dfrac{a+b}{2} \in [0,1]$ then applying this inequality to

$(x,y,p) = \left(\frac{b}{a}, \frac{a}{b}, \sqrt{ab}\right)$ and $(x,y,p) = \left(\frac{b}{a}, \frac{a}{b}, \frac{a+b}{2}\right)$

we obtain $\dfrac{a\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+b\left(\dfrac{a}{b}\right)^{\sqrt{ab}}}{a+b} \le \left(\dfrac{a\cdot\dfrac{b}{a}+b\cdot\dfrac{a}{b}}{a+b}\right)^{\sqrt{ab}} = 1 \iff$

$a\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+b\left(\dfrac{a}{b}\right)^{\sqrt{ab}} \le a+b$ and

$\dfrac{a\left(\dfrac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\dfrac{a}{b}\right)^{\frac{a+b}{2}}}{a+b} \le \left(\dfrac{a\cdot\dfrac{b}{a}+b\cdot\dfrac{a}{b}}{a+b}\right)^{\frac{a+b}{2}} = 1 \iff$

$a\left(\dfrac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\dfrac{a}{b}\right)^{\frac{a+b}{2}} \le a+b.$

Therefore, $a\left(\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\dfrac{b}{a}\right)^{a+b}}\right)+b\left(\left(\dfrac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\dfrac{a}{b}\right)^{a+b}}\right) \le 2\left(a+b\right).$

Solution 4 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY

In the following proof, we use Heinz's inequality:

$$\sqrt{ab} \ \le\ \frac{a^{1-\alpha}b^{\alpha}+a^{\alpha}b^{1-\alpha}}{2} \ \le\ \frac{a+b}{2} \quad \text{for } a,b>0,\ \alpha\in[0,1],$$

first with $\alpha=\sqrt{ab}$ and then with $\alpha=(a+b)/2$.

First we rearrange the central term in the proposed inequality:

$$\begin{aligned}
&a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\\
&= a\left(\frac{b}{a}\right)^{\sqrt{ab}} + a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\\
&= \left(a\cdot\frac{b^{\sqrt{ab}}}{a^{\sqrt{ab}}}+b\cdot\frac{a^{\sqrt{ab}}}{b^{\sqrt{ab}}}\right) + \left(a\cdot\frac{b^{\frac{a+b}{2}}}{a^{\frac{a+b}{2}}}+b\cdot\frac{a^{\frac{a+b}{2}}}{b^{\frac{a+b}{2}}}\right)\\
&= 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right) + 2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right).
\end{aligned}$$

Now the Heinz inequality yields

$$\begin{aligned}
4\sqrt{ab} &\le 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right) + 2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)\\
&\le 2\left(\frac{a+b}{2}\right) + 2\left(\frac{a+b}{2}\right) = 2(a+b).
\end{aligned}$$

Solution 5 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\geq 4\sqrt{ab}. \tag{1}$$

To prove this, we will use the well-known inequality that for all $p>0$ and any real number r

$$p^r+\frac{1}{p^r}\geq 2. \tag{2}$$

For all $a>0$ and $b>0$, using (2) in the above step, we can write

$$\begin{aligned}
& a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\\
= & \sqrt{ab}\left[\sqrt{\frac{a}{b}}\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+\sqrt{\frac{b}{a}}\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\right]\\
= & \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{b}{a}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{a}{b}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]\\
& \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]\\
\geq & \sqrt{ab}(2+2)=4\sqrt{ab}.
\end{aligned}$$

This completes the proof of (1).

Now, we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\leq 2(a+b). \tag{3}$$

To prove (3), we notice that, for $a\in(0,1]$ and $b(0,1]$.

With $a\leq b$, we have $\begin{cases} b^{\sqrt{ab}}-a^{\sqrt{ab}}\geq 0\\ b^{1-\sqrt{ab}}-a^{1-\sqrt{ab}}\geq 0, \end{cases}$ and $\begin{cases} b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\geq 0\\ b^{1-\frac{a+b}{2}}-a^{1-\frac{a+b}{2}}\geq 0 \end{cases}$. (4)

Also,

$$-2a-2b=-a^{\sqrt{ab}}a^{1-\sqrt{ab}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}} \tag{5}$$

Now, using (4) and (5), we have

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)-2a-2b$$

$$= a^{1-\sqrt{ab}}b^{\sqrt{ab}} + a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}} + a^{\sqrt{ab}}b^{1-\sqrt{ab}} + a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$
$$-a^{\sqrt{ab}}a^{1-\sqrt{ab}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$= \left(a^{1-\sqrt{ab}}b^{\sqrt{ab}} - a^{\sqrt{ab}}a^{1-\sqrt{ab}}\right) + \left(a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}\right)$$
$$+ \left(a^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}}\right) + \left(a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}\right)$$

$$= a^{1-\sqrt{ab}}\left(b^{\sqrt{ab}} - a^{\sqrt{ab}}\right) + a^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}}\right)$$
$$-b^{1-\sqrt{ab}}\left(b^{\sqrt{ab}} - a^{\sqrt{ab}}\right) - b^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}}\right)$$

$$= -\left[\left(b^{\sqrt{ab}} - a^{\sqrt{ab}}\right)\left(b^{1-\sqrt{ab}} - a^{1-\sqrt{ab}}\right)\right.$$
$$\left. + \left(b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}}\right)\left(b^{1-\frac{a+b}{2}} - a^{1-\frac{a+b}{2}}\right)\right] \le 0.$$

This completes the proof of (3).

Now, combining the inequalities from (1) an (3), we conclude that

$$4\sqrt{ab} \le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \le 2(a+b)$$

Also solved by Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5621: *Proposed by Stanley Rabinowitz, Brooklyn, NY*

Given; non-negative integer n, real numbers a and c with $ac \neq 0$, and the expression $a+cx^2 \geq 0$.

Express: $\displaystyle\int \left(a+bc^2\right)^{\frac{2n+1}{2}}\,dx$ as the sum of elementary functions.

Solution 1 by Michel Bataille, Rouen, France

We will make use of the following lemma: if z is a nonzero complex number, then